

Phase Transitions in Traffic Models

E. Levine,¹ G. Ziv,¹ L. Gray,² and D. Mukamel¹

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It is suggested that the question of existence of a jamming phase transition in a broad class of single-lane cellular-automaton traffic models may be studied using a correspondence to the asymmetric chipping model. In models where such correspondence is applicable, jamming phase transition does not take place. Rather, the system exhibits a smooth crossover between free-flow and jammed states, as the car density is increased.

KEY WORDS: Driven diffusive systems; traffic jams; condensation transition.

Traffic flow and the formation of traffic jams have been extensively studied for many years.⁽¹⁾ A very useful quantity which has often been used to characterize traffic flow is the relation between the density of cars in the road and the traffic throughput. This relation, termed the *fundamental diagram*, was measured empirically in various situations, and was studied in a large variety of models.^(1,2) When the density of cars in the system is low, the traffic flow is expected to grow linearly with the density. At high densities traffic jams are formed, lowering down the flow sometimes even to a complete stop.^(2,3) This enables one to identify three regimes in the density-flow plane: a free-flow regime at low densities; a regime of wide moving jams at high densities; and a synchronized flow regime, where jams and free-flow coexist, at intermediate densities.⁽¹⁾ The question whether the transition from one regime to another is a smooth crossover or is a result of a genuine phase transition is still not settled in most traffic models.⁽⁴⁾

In recent years Probabilistic Cellular Automata (CA) models have been introduced to analyze traffic flow.^(1,5,6) In such models both time and

¹Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel; e-mail: fnmukaml@wisemail.weizmann.ac.il

²School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA

space are discrete and the physical state of the system (e.g., position and velocity of all cars) is updated simultaneously according to some update scheme. This provides a rather efficient way for carrying out numerical studies of the fundamental diagram. Some traffic CA models were suggested to exhibit phase transitions.^(7–10) However, the existence of such a transition can only be explicitly demonstrated in some limiting cases where certain dynamical processes are deterministic. The existence of a jamming phase transitions in more generic cases, where all dynamical processes are non-deterministic, is still an open question. Such transitions have been suggested to occur in some models on the basis of mean-field methods and numerical simulations, which cannot yield a definitive answer to this question. In this Letter we address this issue in more detail.

Within a more general framework, the existence of phase transitions in one-dimensional driven systems has been studied rather extensively in recent years and several mechanisms for such transitions have been proposed. Examples include the zero range process,^(11,12) two species driven models⁽¹³⁾ and the chipping model.^(14–17) The chipping model incorporates dynamical processes which are closely related to those taking place in traffic dynamics, so one would hope to obtain useful insights for traffic jams from what is known about the chipping model.

In this paper we examine the correspondence between traffic CA models and the chipping model more closely. This correspondence suggests that for a large class of traffic models with non-deterministic dynamics, a genuine phase transition is not expected. Rather, these models exhibit a smooth crossover between the free flow and the jammed phases. In the following we briefly review the main known results for the chipping model. We then introduce a simple CA traffic model for which the correspondence to the chipping model could be made explicit. Other CA traffic models which have been introduced and studied in the past are also examined within this approach.

We start by considering the *Chipping Model* (CM). The model is defined on a periodic lattice, where each site can contain any number of particles. The dynamics is defined through the rates by which two nearest neighbor sites containing k and m particles, respectively, exchange particles:

$$(k, m) \xrightarrow{1} (k+m, 0), \quad (k, m) \xrightarrow{\omega_L} (k+1, m-1), \quad (k, m) \xrightarrow{\omega_R} (k-1, m+1). \quad (1)$$

The first is a diffusion (or coalescence) process³ while the last two processes correspond to right and left chipping of a particle from one site

³Here only a diffusion to the left is considered. One can also introduce diffusion to the right, without changing the relevant results.

to the other. It has been shown in ref. 16 that if the chipping process is symmetric ($\omega_R = \omega_L$) there is a condensation transition at a critical density, above which one site becomes macroscopically occupied. Furthermore, numerical simulations and mean-field studies show that the probability $P(k)$ of finding k particles in a site has the asymptotic form $P(k) \sim z^k/k^\tau$ for large k , with $\tau=3/2$. At the critical point $z=1$ and $\tau=5/2$. The parameter $z \leq 1$ is determined by the average particles density and serves as the fugacity. The condensation transition is a result of the fact that $\tau > 2$, for which the distribution $P(k)$ cannot sustain high densities even at $z=1$. This transition is analogous to the Bose–Einstein condensation. In contrast, if the chipping is asymmetric there exists no phase transition at any density.⁽¹⁷⁾ In this case numerical studies indicate that the domain size distribution has the same form as above, but here $\tau=2$. This distribution remains valid at any density with z approaching 1 at high densities, indicating that no condensation transition takes place.

In the following we argue that the chipping model with an asymmetric chipping process provides a framework within which a large class of traffic models can be characterized. Starting from a particular traffic model we first identify the *domains* which characterize the flow. A domain can either be a low density segment, termed a *gap* or a *hole* in some studies ; a high density segment, termed a *jam* ; or a segment of some other characteristics, defined ad-hoc. A domain of size k is then associated with a site of the CM occupied by k particles. One then proceeds by examining the evolution of the domains, and identifying their dynamical processes. As will be demonstrated, in many cases these processes are closely related to the diffusion and the chipping processes of the asymmetric CM.

We now consider a particular traffic model for which the correspondence to the CM can be made rather explicit. The model, referred as Velocity Dependent Braking (VDB), is a variant of the Nagel–Schreckenberg model.⁽⁶⁾ It is defined on a periodic lattice of size L with $M = \rho L$ cars. Each car is characterized by a velocity $v_i(t) = 0 \dots v_{\max}$ and a position $x_i(t)$. The dynamics is performed in parallel by first updating the velocities as

$$v_i(t+1) = \begin{cases} \min\{v_i(t) + 1, v_{\max}, x_{i+1}(t) - x_i(t) - 1\} & \text{with probability } 1 - p(v_i(t)), \\ 0 & \text{with probability } p(v_i(t)) \end{cases} \quad (2a)$$

and then increasing the position of each car by its speed,

$$x_i(t+1) = x_i(t) + v_i(t+1). \quad (2b)$$

The braking probability $p(v)$ is defined in terms of the two parameters of the model, p and q , as

$$p(v) = \begin{cases} p, & v < v_{\max}, \\ q, & v = v_{\max}. \end{cases} \quad (3)$$

We proceed by first considering the model in the *cruise control* (CC) limit, $q = 0$. In this limit cars residing in dilute regions of the system move deterministically with the maximal allowed velocity v_{\max} . It is thus straightforward to define a free-flow domain as a segment which consists of vacancies and deterministically moving cars. For $\rho \leq \rho_f = 1/(v_{\max} + 1)$ a free-flow steady state exists, where all cars move deterministically and the current is simply given by $J(\rho) = v_{\max} \rho$. At higher densities such a state does not exist, local jams are formed and the current is reduced. A phase transition between the two regimes thus takes place at some density $\rho_0 \leq \rho_f$. This particular phase transition is a result of the fact that due to the deterministic processes, the free-flow state is an absorbing state which has no dynamics. It is thus expected to exist only in the CC limit, and to turn into a smooth crossover for $q > 0$, where no absorbing state exists.

It is straightforward to analyze the fundamental diagram in the CC limit ($q = 0$) for the case $v_{\max} = 1$. We find $\rho_f = 1/2$ and $\rho_0 = (1 - p)/(2 - p)$.⁽¹⁸⁾ It can be shown that in the jammed state ($\rho > \rho_0$) the current is $J(\rho) = \rho_0(1 - \rho)/(1 - \rho_0)$. For $\rho_0 \leq \rho \leq \rho_f$ both free-flow states and jammed states coexist in the thermodynamic limit. In this region a free-flow state evolves deterministically and jams are never produced. However, starting from a random initial condition a jammed state is formed, which slowly evolves towards the free-flow one. However, the time it takes for a system to reach a free-flow state increases exponentially with the system size.⁽¹⁸⁾ Thus in the thermodynamic limit both the free flow and the jammed phases exist as stable steady states of the system. For $q > 0$ the $J(\rho)$ curve can be calculated numerically, and it shows no singularity, as expected. The fundamental diagram of the model for the case $v_{\max} = 1$ is given in Fig. 1a.

As in many traffic models, the general features of the model are revealed only at $v_{\max} > 1$. We still expect a genuine phase transition between the free-flow and jammed states in the CC limit. Again this transition is expected to turn into a crossover for $q > 0$ (see Fig. 1b). The question is whether there is another transition at $\rho > \rho_0$ which is not associated with the existence of absorbing free-flow states, and which may persist beyond the CC limit, namely for $q > 0$.

In what follows we consider $v_{\max} = 2$ in the $\rho > \rho_0$ regime. Here no exact solution is available. Instead, we analyze the evolution of free-flow

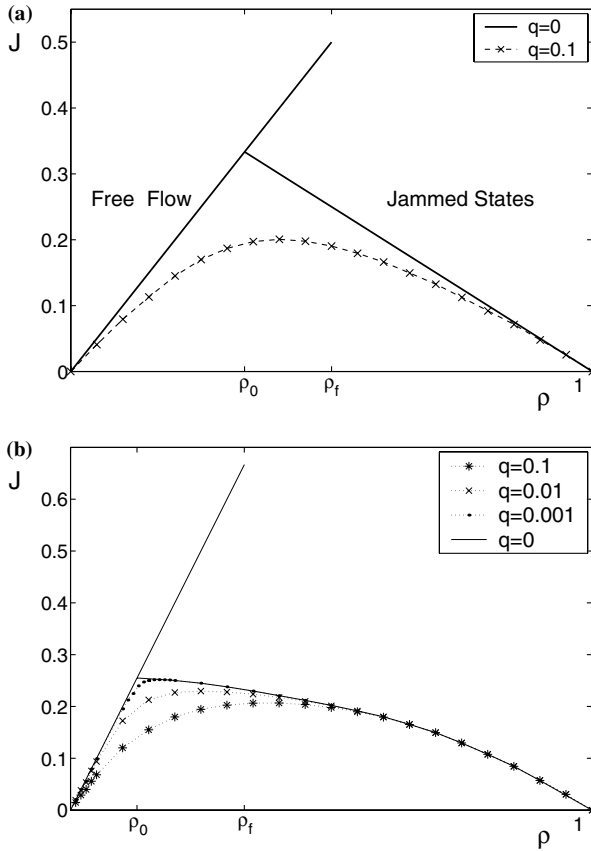


Fig. 1. Fundamental diagram for the VDB model. Data for $q > 0$ was obtained from numerical simulations of systems of size 500. (a) $v_{\max} = 1$ and $p = 1/2$. (b) $v_{\max} = 2$ and $p = 0.6$; here the $q = 0$ free-flow branch is given by 2ρ , while the branch of jammed states in this case was extrapolated from numerical data.

domains, and examine its correspondence to the dynamical processes of the chipping model. In Fig. 2 we present space-time configurations of the model at $v_{\max} = 2$ and $p = 0.1$. Focusing on a single domain (Fig. 2a) we observe that the domain evolves by biased diffusion and chipping processes. In Fig. 2b the evolution of many domains is depicted, demonstrating exchange of particles between domains and coalescence of two domains upon contact.

More explicitly, the dynamics of the domains can be linked to the microscopic processes of the traffic model as follows:

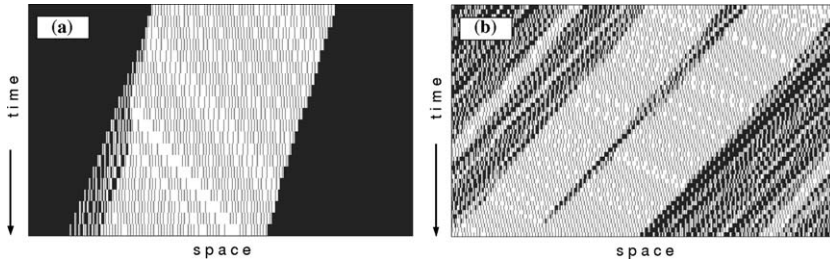


Fig. 2. Space-time configurations of the VDB model in the CC limit, with $p=0.3$. Stopped cars are in black, moving cars in gray, vacancies in white. (a) The evolution of a single domain is characterized by a biased diffusion and chipping of small domains. Here we initiate the model with a single free-flow domain, and present every 5th sweep. (b) Chipping and diffusion may lead to coalescence of neighboring domains. Here we take a random initial condition, and let the system evolve for 1400 sweeps before presenting configurations every 30 sweeps.

1. *Diffusion and Coalescence*—Consider a stationary car, located at the left end of a free-flow domain. This car may accelerate to the maximal velocity v_{\max} , reach the right boundary and eventually stop there. This process corresponds to the diffusion of the whole domain to the left (Fig. 3a).

2. *Chipping*—A stationary car at the left end of a domain may accelerate, and then brake before reaching the maximal velocity. This process decreases the size of the domain, and ejects a single vacancy to the left (Fig. 3b).

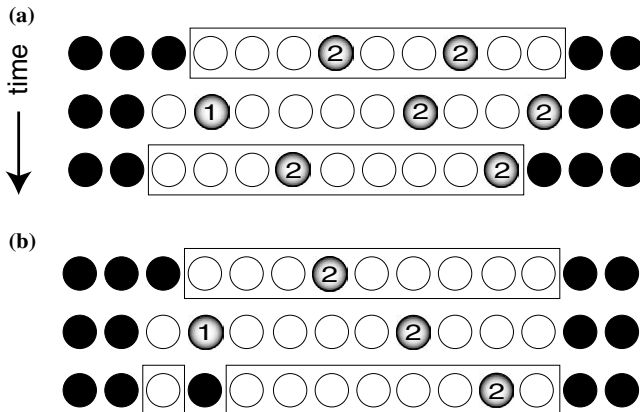


Fig. 3. The $v_{\max} = 2$ dynamics of a domain, depicting (a) diffusion and (b) chipping processes. Stationary cars are marked in black and vacancies in white. The velocity of moving cars, marked in gray, is indicated.

Note that chipping occurs only to the left, corresponding to a fully asymmetric CM, with $\omega_R = 0$.

To test this picture we performed Monte-Carlo simulations of the VDB model with $q = 0$ and $v_{\max} = 2$, and measured the domain size distribution. For convenience we define the size of a free-flow domain, k , as the number of vacancies it contains.⁴ We find that the asymptotic form of $P(k)$ is consistent with $k^{-\tau}$ with $\tau = 2$ for a range of densities $\rho > \rho_0$ and for various values of the parameter p . For example, $P(k)$ for the case $p = 0.3$ and $\rho = 0.4$ is given in Fig. 4. In view of the fact that $\tau = 2$, the existence of a macroscopic domain in this case, as is demonstrated by the peak at large blocks, should be interpreted as a finite-size effect, which would disappear in the thermodynamic limit.⁽¹⁷⁾

It is interesting to note that unlike the CM, the number of domains in the traffic model is not conserved, but is subject to fluctuations. The number of domains is given by $M - M_D$ where M_D is the number of

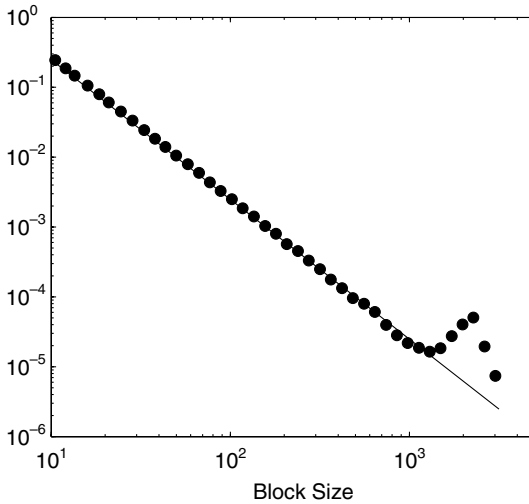


Fig. 4. Domain size distribution of VDB model with $v_{\max} = 2$, $\rho = 0.4$, $p = 0.3$ and $q = 0$. Solid line has a slope -2 . Simulation was performed on a system of size $L = 10,000$ and averaged over 15×10^7 sweeps.

⁴This definition is not unique, as one may include the deterministic cars residing in a domain in the definition of the length, without changing the results. One advantage of the definition used here is that the overall length of domains is conserved.

deterministic cars,⁵ i.e., cars moving at $v = v_{\max}$. The fluctuations in M_D are expected to scale as \sqrt{L} . This was verified by direct numerical studies of systems of size up to 10^4 . Therefore, these fluctuations are not expected to influence the dynamics of the system in the thermodynamic limit.

We conclude that in the CC limit of the VDB model there is no phase transition for $\rho > \rho_0$. It is interesting to examine the implication of this result on the question of existence of a phase transition beyond the CC limit, namely for $q > 0$. Here the free-flow phase is no longer an absorbing state, and thus the transition taking place at ρ_0 in the CC limit is expected to become a smooth crossover. Since no other transition is found at $\rho > \rho_0$ in the CC limit we also expect no such transition for $q > 0$. The reason is that in addition to the CM dynamics, a model with $q > 0$ exhibits other processes, which clearly disfavor condensation of macroscopic domains. For example, here a domain may split into fractions of comparable size, leading to fast fragmentation of large domains.

In examining other traffic models, we find that in many cases the dynamical processes characterizing the CM may still be used to describe the domain dynamics. However, it may turn out that the chipping process involves a detachment of more than a single particle. Nevertheless, as long as the number of chipped particles r is bounded by a finite number, or the probability of chipping r particles $u(r)$ decays sufficiently fast with r (say exponentially), the main results obtained from the CM are expected to be valid. Namely, condensation transition should not take place as long as the chipping process is asymmetric. To verify this point we studied numerically chipping models for the case $u(r) \sim \exp(-r)$ and for the case where r is distributed uniformly over a finite range. In both cases we found that the domain size distribution decays as z^k/k^2 for large k , as expected from the CM. The VDB model with $v_{\max} > 2$ corresponds to CM with chipping of more than a single particle. However, in the CC limit the number of chipped particles is bound by $v_{\max}(v_{\max} - 1)/2$. Thus the analysis presented above for $v_{\max} = 2$ remains valid for any finite v_{\max} .

To demonstrate the more general applicability of the CM picture to traffic models, we briefly consider three other traffic models which have been studied in the past.

Model for Emergent Traffic Jams⁽¹⁹⁾—This is another variant of the Nagel-Schreckenberg traffic model. The model is closely related to the CC limit of the VDB model defined above, except that the velocity update rule

⁵Alternatively, one may associate each domain with a stationary car, say, the one to its left. Within this definition the number of domains is given by the number of stationary cars, where domains of size zero are counted as well. Such a domain corresponds to two adjacent stationary cars. For the purpose of the present study each definition can be used.

(Eq. (2a)) is replaced by

$$\begin{aligned}
 v_i(t + \frac{1}{2}) &= \min \{ v_i(t) + 1, v_{\max}, x_{i+1}(t) - x_i(t) - 1 \}, \\
 v_i(t + 1) &= \begin{cases} v_i(t + \frac{1}{2}) & \text{with probability } 1 - p(v_i(t)), \\ \max \{ 0, v_i(t + \frac{1}{2}) - 1 \} & \text{with probability } p(v_i(t)). \end{cases} \quad (4)
 \end{aligned}$$

Unlike the VDB model, here non-deterministic cars slow down rather than brake, and thus with $v_{\max} = 1$ the two models are identical. This model was shown to exhibit self-organized criticality. Based on general considerations and numerical simulations,⁽¹⁹⁾ it was argued that the size distribution of the domains (termed holes in ref. 19) behaves asymptotically as k^{-2} . Indeed, in ref. 19 the evolution of domains was described in terms similar to those of the CM. Note that in this case the chipping size distribution $u(r)$ decays exponentially with r .

Traffic Cellular Automata⁽²⁰⁾—This model belongs to a different class of CA traffic models where no velocity variable is attached to a car. Cars move to their nearest-neighbor site with a probability that depends on the configuration of their neighborhood. In⁽²⁰⁾ the dynamics is defined as

$$\bullet \bullet \circ \circ \xrightarrow{\alpha} \bullet \circ \bullet \bullet, \quad \circ \bullet \circ \bullet \xrightarrow{\beta} \circ \circ \bullet \bullet, \quad \bullet \bullet \circ \bullet \xrightarrow{\gamma} \bullet \circ \bullet \bullet, \quad \circ \bullet \circ \circ \xrightarrow{\delta} \circ \circ \bullet \bullet, \quad (5)$$

where \bullet denotes a car and \circ a vacancy. In the symmetric CC case, $\gamma = \delta = 1$, the model exhibits a low-density absorbing state at $\rho < 1/3$ and a high-density absorbing state at $\rho > 2/3$. It has been suggested⁽²⁰⁾ that for intermediate densities ($1/3 < \rho < 2/3$) and in some region of the α, β -plane, the system exhibits a macroscopic jam, suggesting a jamming phase transition at some density. The correspondence between this case and the CM is less transparent, and will be addressed in a future publication.⁽²¹⁾ Here we consider the CC limit, $\delta = 1$, with $\gamma < 1$, and apply the CM approach to analyze the jammed phase. In this phase a typical microscopic configuration is given by an alternating left-to-right sequence of (a) free-flow regions, composed of vacancies and cars separated from their nearest neighbor cars by at least two vacancies, (b) finite *mixed* region of alternating cars and vacancies, and (c) an uninterrupted sequence of cars. In order to apply the approach described above, it is convenient to define a domain as a union of adjacent free-flow and mixed regions (a and b above). By examining the dynamics of such domains one finds that they indeed exhibit the characteristic processes of the CM.⁽²¹⁾ We performed Monte-Carlo simulations of this model in the CC limit and measured the

domain size distribution (Fig. 5). We find that the distribution is consistent with k^{-2} , as expected from the CM picture, indicating again that a phase transition does not take place in the jammed state.

It is interesting to note that if instead of $\delta = 1$ we consider $\gamma = 1$, the role played by cars and vacancies is interchanged. Here one can define a domain as a stretch of cars, within which deterministic vacancies are embedded, followed to the left by a mixed region. In this case the distribution of these domains (or jams) behaves as k^{-2} , and thus no macroscopic jam is expected.

Traffic Model with Passing⁽²²⁾—Unlike all models mentioned above, this model is not a cellular automaton. The model is defined on a continuous ring, and evolves in continuous time. Each car is assigned a-priori a random velocity with which it moves on the ring. When a car encounters a slower car it assumes its velocity, thus creating a jam. With some finite probability the next to leading car in a jam can bypass its predecessor and recover its original velocity. The direct correspondence with the CM was already noted in ref 22. Treating the CM within mean-field approximation, the authors concluded that the model should exhibit macroscopic jam. However, numerical simulations show that the jam size distributions is again k^{-2} .⁽²²⁾ Our approach suggests that the observed distribution is

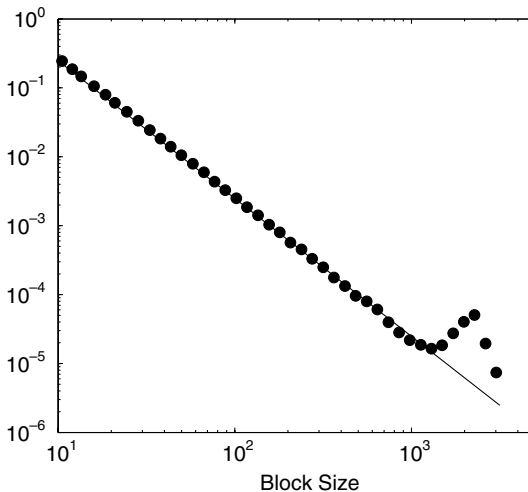


Fig. 5. Domain size distribution of TCA model with $\rho = 0.4$, $\alpha = 0.2$, $\beta = 0.6$, $\gamma = 0.8$ and $\delta = 1$. Solid line has a slope -2 . Simulation was performed on a system of size $L = 10,000$ and averaged over 60×10^6 sweeps.

related to the asymmetric nature of the chipping, a feature which cannot be captured in mean-field.

In summary, it is suggested that in many traffic models the coarse-grained dynamics of domains (of either high or low density) in some deterministic limit, may be described by the two basic processes of the chipping-model, namely diffusion and chipping. Analysis of several traffic models within this approach indicates that as in the asymmetric CM, the traffic models do not exhibit a jamming transition beyond perhaps the one related to the existence of an absorbing state. It is concluded that in non-deterministic traffic models jamming phase transitions do not take place. Rather, a smooth crossover between a free-flow and a jammed state takes place as the car density is increased. The approach outlined in this paper could provide a useful tool for analyzing the behavior of traffic models. In studying a specific model one first has to establish (using numerical or other methods) that indeed the coarse-grained dynamics of the domains does follow the basic processes of the chipping model. Only then one can apply the correspondence between the two. It would be of interest to test the applicability of this approach to broader classes of traffic models.

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